

Kruskal's tree theorem in Type Theory

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Well Quasi Orders (WQO) 1/2

- Important concept in Computer Science:
 - strenghtens well-foundedness, more stable
 - termination of rewriting (Dershowitz, RPO)
 - size-change termination, terminator (Vytiniostis, Coquand ...)
- Important concept in Mathematics:
 - Dickson's lemma, Higman's lemma
 - Higman's theorem, Kruskal's theorem
 - Robertson-Seymour theorem (graph minor theorem)
 - Undecidability result: Kruskal theorem not in PA (Friedman)

Well Quasi Orders (WQO) 2/2

- for \leq a quasi order over X : reflexive & transitive binary relation
- several **classically** equivalent definitions (see e.g. JGL 2013)
 - almost full: each $(x_i)_{i \in \mathbb{N}}$ has a *good pair* ($x_i \leq x_j$ with $i < j$)
 - \leq well-founded and no ∞ antichain
 - finite basis: $U = \uparrow U$ implies $U = \uparrow F$ for some finite F
 - $\{\downarrow U \mid U \subseteq X\}$ well-founded by \subset
- many of these equivalences **do not hold** intuitionistically

WQOs are stable under type constructs

- Given a WQO \leq on X , we can lift \leq to WQOs on:

Higman lemma: $\text{list}(X)$ with $\text{subword}(\leq)$

Higman thm: $\text{btree}(k, X)$ with $\text{emb_product}(\leq)$ (any $k \in \mathbb{N}$)

Kruskal theorem: $\text{tree}(X)$ with $\text{emb_homeo}(\leq)$

- These theorems are *closure properties* of the class of WQOs
- Other noticeable results:

Dickson's lemma: (\mathbb{N}^k, \leq) is a WQO

Finite sequence thm: $\text{list}(\mathbb{N})$ WQO under $\text{subword}(\leq)$

Ramsey theorem: \leq_1 and \leq_2 WQOs imply $\leq_1 \times \leq_2$ WQO

What Intuitionistic Kruskal Tree Theorem?

- The meaning of those closure theorems intuitionistically:
 - depends of what is a WQO (which definition?)
 - but not on e.g. `emb_homeo` which has an inductive definition
- What is a suitable intuitionistic definition of WQO ?
 - quasi-order does not play an important/difficult role
 - should be classically equivalent to the usual definition
 - should intuitionistically imply almost full
 - intuitionistic WQOs must be stable under liftings
- Allow the proof and use of Ramsey, Higman, Kruskal... results

Intuitionistic formulations of WQOs 1/2

- Almost full relations (Veldman&Bezem 93)
 - each $(x_i)_{i \in \mathbb{N}}$ has $x_i R x_j$ with $i < j$
 - works for Higman and Kruskal theorems (Veldman 04)
 - uses *stumps* over \mathbb{N} which require *Brouwer's thesis*
- Bar induction (Coquand&Fridlender 93)
 - Bar (good R) []
 - works for the general Higman lemma (Fridlender 97)
- Well-foundedness (Seisenberger 2003)
 - \ll is well-founded on $\text{Bad}(R)$; $x \ll y$ iff $x = a :: y$ for some a
 - works for Higman lemma and Kruskal theorem
 - requires *decidability* of R

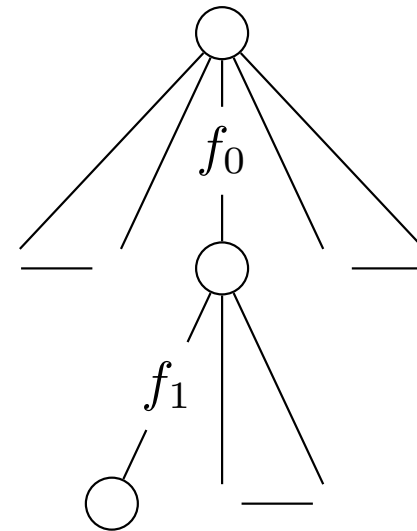
Intuitionistic formulations of WQOs 2/2

- Almost full relations (Vytiniostis&Coquand&Wahlstedt 12)
 - $\text{Af}(R)$ inductively defined
 - works for Ramsey theorem
 - intuitionistically equivalent to Bar (good R) []
- Seisenberger's definition not equiv. to Coquand&Fridlender for undecidable R
- Veldman&Bezem definition works for R over \mathbb{N} (not over arbitrary types) but requires Brouwer's thesis
- Let us introduce Coquand et al. definition

Well-founded trees over a type X

- Well-founded trees $\mathbf{wft}(X)$
 - branching indexed by X
 - the least fixpoint of $\boxed{\mathbf{wft}(X) = \{\star\} + X \rightarrow \mathbf{wft}(X)}$
- Given a branch $f : \mathbb{N} \rightarrow X$, compute its height:

- $f(1 + \cdot) = x \mapsto f(1 + x)$
- $\mathbf{ht}(\mathbf{inl} \star, -) = 0$
- $\mathbf{ht}(\mathbf{inr} g, f) = 1 + \mathbf{ht}(g(f_0), f(1 + \cdot))$



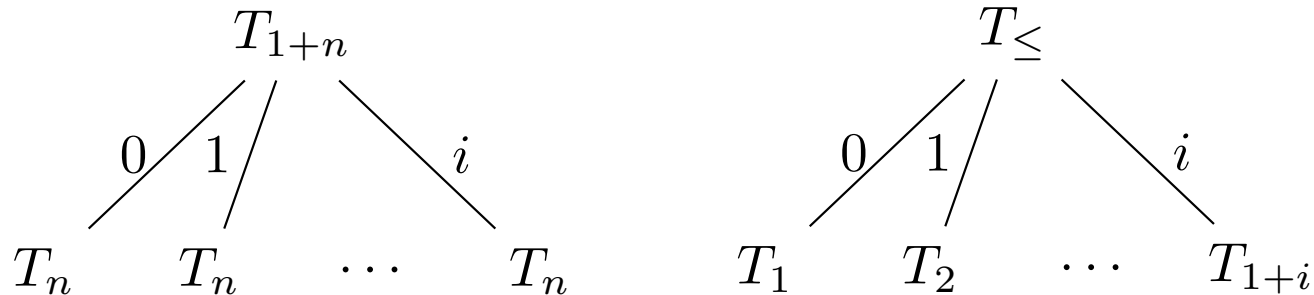
- Veldman's stumps are sets of branches of trees in $\mathbf{wft}(\mathbb{N})$

Coquand's Almost full relations, step by step

1. Veldman et al.: $\forall f : \mathbb{N} \rightarrow X, \exists i < j, f_i R f_j$
2. Logically eq. variant: $\forall f : \mathbb{N} \rightarrow X, \exists n, \exists i < j < n, f_i R f_j$
3. Partially informative: $\forall f : \mathbb{N} \rightarrow X, \{n \mid \exists i < j < n, f_i R f_j\}$
4. Variant: $\{h : (\mathbb{N} \rightarrow X) \rightarrow \mathbb{N} \mid \forall f, \exists i < j < h(f), f_i R f_j\}$
5. Variant: $\{t : \mathbf{wft}(X) \mid \forall f, \exists i < j < \mathbf{ht}(t, f), f_i R f_j\}$
6. Coquand et al.: is defined as an inductive predicate $\mathbf{af_t}(R)$
 - the prefix of length $\mathbf{ht}(t, f)$ of $f : \mathbb{N} \rightarrow X$ contains a good pair
 - the *computational content* is (for every sequence $f : \mathbb{N} \rightarrow X$):
 - a bound on the size of the search space for good pairs
 - and it is not a good pair

A well-founded tree for (\mathbb{N}, \leq)

- Property: $\forall f : \mathbb{N} \rightarrow \mathbb{N}, \exists i < j < 2 + f_0, f_i \leq f_j$
- In $\mathbf{wft}(\mathbb{N})$, we define T_n the tree of uniform height n :
 - $T_0 = \mathbf{inl}(\star)$ and $T_{1+n} = \mathbf{inr}(- \mapsto T_n)$
 - for any $f : \mathbb{N} \rightarrow \mathbb{N}$, $\mathbf{ht}(T_n, f) = n$
- And $T_{\leq} = \mathbf{inr}(n \mapsto T_{1+n})$



- Hence $\mathbf{ht}(T_{\leq}, f) = 1 + \mathbf{ht}(T_{1+f_0}, f(1 + \cdot)) = 2 + f_0$

Almost full relations, inductively

- Lifted relation: $x (R \uparrow u) y = x R y \vee u R x$
 - in $R \uparrow u$, elements above u are forbidden in bad sequences
- $\text{full} : \text{rel}_2 X \rightarrow \boxed{\text{Prop}}$ and $\text{af_t} : \text{rel}_2 X \rightarrow \boxed{\text{Type}}$

$$\frac{\forall x, y, x R y}{\text{full } R} \quad \frac{\text{full } R}{\text{af_t } R} \quad \frac{\forall u, \text{af_t}(R \uparrow u)}{\text{af_t } R}$$

- $\text{af_securedby} : \text{wft}(X) \rightarrow \text{rel}_2 X \rightarrow \text{Prop}$:
 - $\text{af_securedby}(\text{inl } \star, R) = \text{full } R$
 - $\text{af_securedby}(\text{inr } g, R) = \forall u, \text{af_securedby}(g(u), R \uparrow u)$
- these are intuitionistically “equivalent” (hold in `Type`, not `Prop`):
 - $\text{af_t } R$ and $\{t : \text{wft}(X) \mid \text{af_securedby}(t, R)\}$
 - and $\{t : \text{wft}(X) \mid \forall f, \exists i < j < \text{ht}(t, f), f_i R f_j\}$

Almost full relations, by bar inductive predicates

- $\text{good } R : \text{list } X \rightarrow \text{Prop}$
 - $\text{good } R \ ll$ iff $ll = l ++ \boxed{b} :: m ++ \boxed{a} :: r$ for some $a R b$
 - beware of the (implicit) use snoc lists
 - good has an easy inductive definition
- for $P : \text{list } X \rightarrow \text{Prop}$, we define $\text{bar_t } P : \text{list } X \rightarrow \text{Type}$

$$\frac{P \ ll}{\text{bar_t } P \ ll} \qquad \frac{\forall u, \text{bar_t } P \ (u :: ll)}{\text{bar_t } P \ ll}$$

- we show: $\text{af_t}(R \uparrow a_n \uparrow \dots \uparrow a_1)$ iff $\text{bar_t} (\text{good } R) [a_1, \dots, a_n]$
- another characterization: $\boxed{\text{af_t } R \text{ iff bar_t } (\text{good } R) []}$

Almost full relations, some properties

- `af_t_refl`: if `af_t R` then $=_X \subseteq R$ (iff in case X is finite)
- `af_t_inc`: if $R \subseteq S$ and `af_t R` then `af_t S`
- `af_t_surjective` (DLW, easy but very useful):
 - for $f : X \rightarrow Y \rightarrow \text{Prop}$, $R : \text{rel}_2 X$ and $S : \text{rel}_2 Y$
 - if f surjective: $\forall y, \{x \mid f x y\}$
 - if f morphism: $f x_1 y_1$ and $f x_2 y_2$ and $x_1 R x_2$ imply $y_1 S y_2$
 - then `af_t R` implies `af_t S`
- Ramsey (Coquand): `af_t R` and `af_t S` imply `af_t(R ∩ S)`
 - he deduces `af_t(R × S)` and `af_t(R + S)`
- *I stop because you may be almost full (but it is a MUST READ)*

Higman lemma and the subword relation

- Given $R : \text{rel}_2 X$ over a type X
- The subword relation $<_R^w : \text{rel}_2 (\text{list } X)$ defined by 3 rules

$$\frac{}{[] <_R^w []} \quad \frac{l <_R^w m}{l <_R^w b :: m} \quad \frac{a R b \quad l <_R^w m}{a :: l <_R^w b :: m}$$

- also write **subword** R for $<_R^w$
- Higman lemma (Fridlender 97, non informative version):

$$\text{bar (good } R) [] \text{ implies bar (good (subword } R)) []$$

- Nearly the same proof works for **bar_t** instead of **bar**
- But this proof cannot be generalized to finite trees...

The product tree embedding, Higman theorem

- trees with same type for all arities: $\text{tree } X = X \times \text{list}(\text{tree } X)$
- trees of breadth bounded by $k \in \mathbb{N}$:

$$\text{btree } k \ X = \{t \mid \text{tree_fall } (\langle _ | ll \rangle \mapsto \text{length } ll < k) \ t\}$$

- any $t \in T$ is $t = \langle x | t_1, \dots, t_n \rangle$ with $n < k$, $x \in X$ and $t_i \in T$
- for a relation $R : \text{rel}_2 \ X$, we define (needs some work...)

$$\frac{s <_R^\times t_i}{s <_R^\times \langle x_n | t_1, \dots, t_n \rangle} \quad \frac{x \ R \ y \quad s_1 <_R^\times t_1, \dots, s_n <_R^\times t_n}{\langle x | s_1, \dots, s_n \rangle <_R^\times \langle y | t_1, \dots, t_n \rangle}$$

- also write $\text{emb_tree_product } R$ for $<_R^\times$
- Higman thm. (DLW): $\text{af_t } R$ implies $\text{af_t}(<_R^\times)$ on $\text{btree } k \ X$

The homeomorphic embedding, Krukal theorem

- one type X for all arities: $\text{tree } X = X \times \text{list}(\text{tree } X)$
- for $R : \text{rel}_2 X$, we define $<_R^*$ by nested induction

$$\frac{s <_R^* t_i}{s <_R^* \langle x_n | t_1, \dots, t_n \rangle}$$

$$\frac{x_i R x_j \quad [s_1, \dots, s_i] \text{ (subword } <_R^*) [t_1, \dots, t_j]}{\langle x_i | s_1, \dots, s_i \rangle <_R^* \langle x_j | t_1, \dots, t_j \rangle}$$

- ω -continuity to build $<_R^*$ and prove the elimination scheme
- we also write $\text{emb_tree_homeo } R$ for $<_R^*$
- Kruskal theorem (DLW): $\text{af_t } R$ implies $\text{af_t}(<_R^*)$

Plan of the rest of the presentation

- high level and informal proof principles of Higman's theorem
 - with ideas from Veldman (mostly), Fridlender and Coquand
 - $\mathbf{tree}(X_n)_{n < k}$, one type (and one relation) for each arity
- focus on several implementation challenges of that proof
 - $\mathbf{tree}(X_n)$ as a (decidable) subtype of $\mathbf{tree}(\sum X_n)$
 - embed $\sum X_n$ in a (specialized) universe U
 - empty type grounded induction for $\mathbf{af_t}$, ...
- what about the non-informative case \mathbf{af} ?
 - beware $\mathbf{af} R$ is weaker than $\mathbf{inhabited}(\mathbf{af_t} R)$
 - well-foundedness upto a projection
- from Higman theorem to Kruskal theorem (remarks)

The product tree embedding, Higman theorem

- $\text{tree}(X_n)_{n < k} = T$ where T is lfp of $T = \sum_{n=0}^{k-1} X_n \times T^n$
- one type X_n for each arity $n < k$
- any $t \in T$ is $t = \langle x_n | t_1, \dots, t_n \rangle$ with $x_n \in X_n$ and $t_i \in T$
- for arity-indexed relations $R : \forall n < k, \text{rel}_2(X_n)$, we define

$$\frac{s <_R^h t_i}{s <_R^h \langle x_n | t_1, \dots, t_n \rangle} \quad \frac{x_n R_n y_n \quad s_1 <_R^h t_1, \dots, s_n <_R^h t_n}{\langle x_n | s_1, \dots, s_n \rangle <_R^h \langle y_n | t_1, \dots, t_n \rangle}$$

- also write $\text{emb_tree_higman } R$ for $<_R^h$
- Higman thm.: $(\forall n < k, \text{af_t } R_n)$ implies $\text{af_t}(<_R^h)$

Higman theorem, based on (Veldman 2004)

- each $\text{af_t } R_n$ is witnessed by w_n : $\text{af_securedby}(w_n, R_n)$
- *easier* outermost induction on $[w_0, \dots, w_{k-1}]$ (lexicographic)
- apply rule 2, hence prove: $\forall t, \text{af_t} (\langle \! \! \! \langle^h_R \uparrow t \rangle \! \! \! \rangle)$
- do this by structural induction on t
 - $t = \langle x_i | t_1, \dots, t_i \rangle$ with $i < k$
 - we can assume $\text{af_t} (\langle \! \! \! \langle^h_R \uparrow t_1 \rangle \! \! \! \rangle), \dots, \text{af_t} (\langle \! \! \! \langle^h_R \uparrow t_i \rangle \! \! \! \rangle)$
 - we show $\text{af_t} (\langle \! \! \! \langle^h_R \uparrow \langle x_i | t_1, \dots, t_i \rangle \rangle \! \! \! \rangle)$
 - depends on $i = 0$ or not, $w_i = \text{inl} \star$ or not

Higman thm, case of leaves ($i = 0$ and $w_0 = \text{inr } g$)

- we have $t = \langle x_0 | \emptyset \rangle$ ($i = 0$)
- $R'_0 = R_0 \uparrow x_0$ is **af_t**, witnessed by $w'_0 = g(x_0)$
- $R'_j = R_j$ and $w'_j = w_j$ for $0 < j < k$
 - $w'_0 = g(x_0)$ is a sub-**wft**(X_0) of $w_0 = \text{inr } g$, hence simpler
 - $[w'_0, w_1, \dots, w_{k-1}]$ easier than $[w_0, w_1, \dots, w_{k-1}]$
 - we deduce **af_t**($\langle^h_{R'} \rangle$) by induction
- we show $\langle^h_{R'} \rangle \subseteq \langle^h_R \rangle \uparrow \langle x_0 | \emptyset \rangle$ (relatively easy to check)
- we conclude **af_t**($\langle^h_R \rangle \uparrow \langle x_0 | \emptyset \rangle$)

Higman thm, case of leaves ($i = 0$ and $w_0 = \text{inl } \star$)

- $t = \langle x_0 | \emptyset \rangle$
- $R_0 \uparrow x_0 = R_0$ because R_0 is (already) full ($w_0 = \text{inl } \star$)
- but then we have $x_0 R_0 y$ for any y
- hence we deduce $\langle x_0 | \emptyset \rangle <_R^h \langle x_j | v_1, \dots, v_j \rangle$
 - any (finite) tree contains a leaf $\langle y | \emptyset \rangle$
 - $\langle x_0 | \emptyset \rangle$ embeds into any leaf, e.g. $\langle y | \emptyset \rangle$
- we deduce $<_R^h \uparrow \langle x_0 | \emptyset \rangle$ is full (trivial to check)
- we conclude $\mathbf{af_t}(<_R^h \uparrow \langle x_0 | \emptyset \rangle)$

Higman thm ($0 < i < k$ and $w_i = \text{inr } g$) 1/2

- let $T = \text{tree}(X_0, \dots, X_{k-1})$
- we have $t = \langle x_i | t_1, \dots, t_i \rangle$ with $0 < i < k$
- $X'_j = X_j$ and $R'_j = R_j$ for $j \notin \{i-1, i\}$
- $X'_i = X_i$ and $R'_i = R_i \uparrow x_i$ is **af_t** for $w'_i = g(x_i)$ simpler than w_i
- $X'_{i-1} = X_{i-1} + \sum_{p=0}^{i-1} X_i \times T$ and $R'_{i-1} = R_{i-1} + \sum_{p=0}^{i-1} R_i \times (\langle^h_R \uparrow t_p)$
 - R'_{i-1} is **af_t** by Ramsey, obtain w'_{i-1}
 - $[\dots, w'_{i-1}, w'_i, \dots]$ easier than $[\dots, w_{i-1}, w_i, \dots]$
 - we deduce **af_t**($\langle^h_{R'}$) by induction
- we show **af_t**($\langle^h_{R'}$) implies **af_t**($\langle^h_R \uparrow \langle x_i | t_1, \dots, t_i \rangle$) (not easy)

Higman thm ($0 < i < k$ and $w_i = \text{inr } g$) 2/2

- with $X'_{i-1} = X_{i-1} + \sum_{p=0}^{i-1} X_i \times T$, define an *evaluation map*
- $\text{ev} : \text{tree}(X_0, \dots, X'_{i-1}, X_i, \dots) \rightarrow \text{tree}(X_0, \dots, X_{k-1})$
 - $\text{ev}(\langle y_j | t_1, \dots, t_j \rangle) = \langle y_j | \text{ev } t_1, \dots, \text{ev } t_j \rangle$ for $j \neq i - 1$
 - $\text{ev}(\langle y_{i-1} | t_1, \dots, t_{i-1} \rangle) = \langle y_{i-1} | \text{ev } t_1, \dots, \text{ev } t_{i-1} \rangle$
 - $\text{ev}(\langle (p, y_i, t) | t_1, \dots, t_{i-1} \rangle) = \langle y_i | \text{insert } t \text{ } p [\text{ev } t_1, \dots, \text{ev } t_{i-1}] \rangle$
- ev (is surjective and) has finite inverse images
 - allows the use of `bar_t` induction and the FAN theorem
- use ev to show $\text{af_t}(\langle^h_{R'} \rangle)$ implies $\text{af_t}(\langle^h_R \uparrow \langle x_i | t_1, \dots, t_i \rangle \rangle)$
 - combinatorial principle: $\forall x \in X, P_x \vee Q_x \Rightarrow \forall x P_x \vee \exists x Q_x$
 - and more complex version (see later)
 - very technical part of Coq proof (largely absent from paper)

Higman thm ($0 < i < k$ and $w_i = \text{inl } \star$)

- $T = \text{tree}(X_0, \dots, X_{k-1})$ and $t = \langle x_i | t_1, \dots, t_i \rangle$ with $0 < i < k$
- $w_i = \text{inl } \star$ thus we have R_i is full on X_i
- X'_j and R'_j for $j \neq i$ as in case $w_i = \text{inr } g$
- $X'_i = \emptyset$ with any R'_i (only one exists) is `af_t`
- ensure case where $X'_i = \emptyset$ is simpler than R_i is full on X_i
 - $w'_i = \text{None}$ is simpler than $w_i = \text{Some}(\text{inl } \star)$
 - we deduce `af_t`($\langle^h_{R'} \rangle$) by induction
- we show `af_t`($\langle^h_{R'} \rangle$) implies `af_t`($\langle^h_R \uparrow \langle x_i | t_1, \dots, t_i \rangle \rangle$)
 - similar to the case $w_i = \text{inr } g$
 - but not easy to factorize the Coq duplicated code

Higman thm ($i < k$ and $w_i = \text{None}$)

- $T = \text{tree}(X_0, \dots, X_{k-1})$ and $t = \langle x_i | t_1, \dots, t_i \rangle$ with $0 < i < k$
- but because $w_i = \text{None}$, we have $X_i = \emptyset$
- this contradicts $x_i \in X_i$; an easy case indeed

The induction principle of Veldman's proof

- lexicographic product (corresponds to nested induction)
- not grounded on full relations (witnessed by the empty `wft`)
- but grounded on empty types
- empty types are sub-cases of full relations

Remarks on the implementation of that proof

- Implements “well” for e.g. at most unary/binary trees

Theorem `higman_abt_t` : forall Z T, @af_t Z T
-> forall Y S, @af_t Y S
-> forall X R, @af_t X R
-> af_t (embed_abtree R S T).

Proof. do 3 (induction 1 using af_t_dep_rect); End.

- Thought it requires a dependent induction principle for `af_t`
- But that does not work for parameterized breadth k
 - `tree(X_n) $_{n < k}$` VERY cumbersome to work with
 - `[... , w'_{i-1} , w'_i , ...]` “easier” than `[... , w_{i-1} , w_i , ...]`
 - but the `$w'_{i-1} : \text{wft } X'_{i-1}$` and `$w_{i-1} : \text{wft } X_{i-1}$` not same type !!

A dependent induction principle for af_t

Section af_t_dep_rect.

Variable (P : forall X, relation X -> Type).

Hypothesis HP0 : P ER.

Hypothesis HP1 : forall X R, full R -> P ER -> @P X R.

Hypothesis HP2 : forall X R, (forall x, af_t (R rlift x))
-> (forall x, P (R rlift x))
-> @P X R.

Theorem af_t_dep_rect : forall X R, af_t R -> @P X R.

End af_t_dep_rect.

Finite Trees in Coq

- Dependent types: nice way to represent complex data structures
- But too much dependency can make your life miserable
- Hence we represent $\text{tree}(X_n)_{n \in \mathbb{N}}$ by:

$$\{t : \text{tree}(\sum X_n) \mid \text{tree_fall } (x \ ll \mapsto \text{arity } x = \text{length } ll) \ t\}$$

- $\text{tree } X$ is the lfp of $\text{tree } X = X \times \text{list}(\text{tree } X)$:

Variable X : Type.

Inductive tree : Type := in_tree : X -> list tree -> tree.

- Can freely use the `List` library to deal with the forest of sons
- Nested definition does not generate a good elimination scheme

Finite Trees in Coq, a nice recursor

Variable P : tree -> Type.

Hypothesis f : forall a ll, (forall x, In_t x ll -> P x)
-> P (in_tree a ll).

Fixpoint tree_rect t : P t :=
 match t with
 | in_tree a ll => f a ll (map_t P tree_rect ll)
 end.

Hypothesis f_ext : ...

Fact tree_rect_fix a ll : tree_rect (in_tree a ll)
 = f a ll (fun t _ => tree_rect t).

Finite trees in Coq, example definitions

```
Implicit Types (P : X -> list tree -> Prop)
              (Q : nat -> X -> Prop).
```

```
Definition tree_fall P : tree -> Prop.
```

```
Fact tree_fall_fix P x ll :
  tree_fall P (in_tree x ll)
  <-> P x ll
  /\ forall t, In t ll -> tree_fall P t.
```

```
Let btree k := tree_fall (fun x ll => length ll < k).
```

```
Let wfptree Q := tree_fall (fun x ll => Q (length ll) x).
```

Higman Embedding in Coq

Variables (X : Type) (R : nat -> X -> X -> Prop).

```
Inductive emb_tree_higman : tree X -> tree X -> Prop :=
  | in_emb_tree_higman_0 : forall s t x ll,
    In t ll
    -> s <eh t
    -> s <eh in_tree x ll
  | in_emb_tree_higman_1 : forall x y ll mm,
    R (length ll) x y
    -> Forall2 emb_tree_higman ll mm
    -> in_tree x ll <eh in_tree y mm
where "x <eh y" := (emb_tree_higman x y).
```

Higman Embedding in Coq, elimination Scheme

Variable S : tree X -> tree X -> Prop.

Infix "<<" := S (at level 70).

Hypothesis S_sub0 : forall s t x ll,
 In t ll -> s <eh t
 -> s << t -> s << in_tree x ll.

Hypothesis S_sub1 : forall x y ll mm, R (length ll) x y
 -> Forall2 emb_tree_higman ll mm
 -> Forall2 S ll mm
 -> in_tree x ll << in_tree y mm.

Theorem emb_tree_higman_ind t1 t2 : t1 <eh t2 -> t1 << t2.

Almost Full predicate

Definition af_t R := { t : wft X | af_securedby R t }.

Inductive af_type : (X -> X -> Prop) -> Type :=
| in_af_type0 : forall R, full R -> af_type R
| in_af_type1 : forall R, (forall a, af_type (R rlift a))
-> af_type R.

Definition af_t_other R :=
{ t | forall f, good R (pfx_rev f (wft_ht t f)) }.

Thm af_t_eq : af_t R <-> af_type R <-> af_t_other R.

Inductive Bar predicates

```
Implicit Types (P : list X -> Prop) (R : X -> X -> Prop).
```

```
Inductive bar_t P : list X -> Type :=  
  | in_bar_t0 : forall ll, P ll -> bar_t P ll  
  | in_bar_t1 : forall ll, (forall a, bar_t P (a::ll))  
    -> bar_t P ll.
```

```
Inductive good R : list X -> Prop :=  
  | in_good_0 : forall ll a b, In b ll  
    -> R b a -> good R (a::ll)  
  | in_good_1 : forall ll a, good R ll -> good R (a::ll).
```

```
Thm af_t_bar_t R : af_t R <-> bar_t (good R) nil.
```

A universe tailored for Higman theorem

- Given a type $(X_i)_{i < k}$, a universe U is a post fixpoint of:

$$U = \{\star\} + \sum X_i + U + \mathbb{N} \times U \times \mathbf{tree} U$$

- Then $X'_{i-1} = X_{i-1} + \sum_{p=0}^{i-1} X_i \times \mathbf{tree}(X_0, \dots, X_{k-1})$ can be viewed as a sub-type of U (in Veldman 2004, $U = \mathbb{N}$)

Variable X : Type.

Inductive htree_fix :=

```
| in_htf_u : htree_fix          (* undefined *)
| in_htf_0 : X -> htree_fix     (* X embeds in U *)
| in_htf_1 : htree_fix -> htree_fix (* U embed in U *)
| in_htf_2 : nat -> htree_fix
          -> tree htree_fix -> htree_fix.
```

Higman theorem, the recursive statement

Definition owft X := option (wft X).

Variables (X : Type) (k : nat).

Notation U := (htree_fix X).

Theorem higman_htree_rec (s : nat -> owft U) :

```
  forall P : nat -> U -> Prop,
    (forall n, ~ P n (@in_ktf_u X))
-> (forall n x, { P n x } + { ~ P n x })
-> (forall n, k < n -> P n = fun _ => False)
-> forall R,
    (forall n, n <= k -> afs_owft_sec (s n) (P n) (R n))
-> afs_t (wfptree P) (emb_tree_higman R).
```

What about the logical version

- $\text{af} : \text{rel}_2 X \rightarrow \text{Prop}$ instead of $\text{af_t} : \text{rel}_2 X \rightarrow \text{Type}$
- Unlike provable/has a proof, $\text{af } R$ is NOT $\text{inhabited}(\text{af_t } R)$
 - cannot use empty `wft` to decide when R is full or not !!
- To get $\text{af } R \Rightarrow \exists t, \text{af_securedby}(t, R)$, you either need:
 - `FunctionalChoice_on`: $(\forall x \exists y, x R y) \Rightarrow \exists f \forall x, x R f(x)$
 - or *Brouwer's thesis* (Veldman 2004)
- How to replace lexicographic induction of `wft` sequences?
 - first idea: encode lex. product at `Prop` level instead of `Type`
 - new idea: use well-founded upto relations

wf. upto rels. are stable under lex. products

Well-founded upto relations

```
Variable (X Y : Type).
```

```
Implicit Type (f : X -> Y) (R : relation X)
              (P : X -> Prop) (Q : Y -> Prop).
```

```
Definition well_founded R :=
  forall P, (forall a, (forall b, R b a -> P b)
                -> P a)
            -> forall a, P a.
```

```
Definition well_founded_upto f R :=
  forall Q, (forall a, (forall b, R b a -> Q (f b))
                -> Q (f a))
            -> forall a, Q (f a).
```

Almost full rels and Wf upto 1/2

Inductive afw : Set := af_empty | af_full | af_rlift.

Let lt_afw : afw -> afw -> Prop. (* empty < full < rlift *)

Definition af_subrel := (af_w * ((X -> Prop) * relation X)).

Definition afsr_correct (c : af_subrel) :=

 match c with

 | (af_empty, (P, _)) => forall x, ~ P x

 | (af_full , (P,R)) => forall x y, P x -> P y -> R x y

 | (af_rlift, (P,R)) => afs P R

 end.

Almost full rels and Wf upto 2/2

```
Definition lt_afsr (c1 c2 : af_subrel) :=
  match c1          , c2 with
  | (w1,(P1,R1)) , (w2,(P2,R2))
  => lt_afw w1 w2
  \/\ w1 = w2
  /\ w1 = af_rlift
  /\ P1 = P2
  /\ exists p, P1 p
  /\ R1 = (R2 rlift p)
  end. (* this relation has reflexive elements *)
```

```
Theorem lt_afsr_upto_wf :
  well_founded_upto (@snd _ _) afsr_correct lt_afsr.
```


What about Kruskal's tree theorem ?

- Shares the same structure as Higman theorem
- There are twice as many cases
- The proof uses both Higman lemma and Higman theorem
- The lexicographic product is a bit different: *more facile*
- The universe is not the same:

$$U = \{\star\} + X + U + U \times \text{list}(\text{list}(\text{tree } U))$$

- Replace `insert` with the more general `intercalate`

$$\text{intercalate } [a_1, \dots, a_n] [l_0, \dots, l_n] = l_0 ++ a_1 :: \dots ++ a_n :: l_n$$

- `emb_tree_upto`: inbetween the product and the homeomorphic

Tree Embedding upto k

- $\text{tree}(X_n)_{n \in \mathbb{N}} = T$ where T is lfp of $T = \sum_{n=0}^{\infty} X_n \times T^n$
- $k \in \mathbb{N}$ and an arity-indexed relation $R : \forall n \in \mathbb{N}, \text{rel}_2(X_n)$
- one X_n for each arity, but $X_k = X_n$ as soon as $n \geq k$

$$\begin{array}{c}
 s <_{k,R}^u t_i \\
 \hline
 s <_{k,R}^u \langle x_n | t_1, \dots, t_n \rangle \\
 \\
 n < k \quad x_n R_n y_n \quad s_1 <_{k,R}^u t_1, \dots, s_n <_{k,R}^u t_n \\
 \hline
 \langle x_n | s_1, \dots, s_n \rangle <_{k,R}^u \langle y_n | t_1, \dots, t_n \rangle \\
 \\
 k \leq i \quad x_i R_k x_j \quad [s_1, \dots, s_i] \text{ (subword } <_{k,R}^u \text{)} [t_1, \dots, t_j] \\
 \hline
 \langle x_i | s_1, \dots, s_i \rangle <_{k,R}^u \langle x_j | t_1, \dots, t_j \rangle
 \end{array}$$

Coq code for `emb_tree_upto`

```
Variables (k : nat) (R : nat -> X -> X -> Prop).
Inductive emb_tree_upto : tree X -> tree X -> Prop :=
  | in_embut_0 : forall s t x ll, In t ll -> s <eu t
    -> s <eu in_tree x ll
  | in_embut_1 : forall x y ll mm, length ll < k
    -> R (length ll) x y
    -> Forall2 emb_tree_upto ll mm
    -> in_tree x ll <eu in_tree y mm
  | in_embut_2 : forall x y ll mm, k <= length ll
    -> R k x y
    -> subword emb_tree_upto ll mm
    -> in_tree x ll <eu in_tree y mm
where "x <eu y" := (emb_tree_upto x y).
```

Kruskal's Tree Theorem, the recursive statement

Variables (X : Type).

Notation U := (ktree_fix X).

Theorem kruskal_ktree_rec (s : nt_stump U) :

```
  forall k, k = nts_char s
-> forall (P : nat -> U -> Prop),
      (forall n, ~ P n in_ktf_u)
      -> (forall n x, { P n x } + { ~ P n x })
      -> (forall n, k <= n -> P k = P n)
-> forall (R : nat -> relation U),
      (forall n, n <= k
        -> afs_owft_sec (nts_seq s n) (P n) (R n))
-> afs_t (wfptree P) (emb_tree_upto k R).
```